

Polarization effects in radiative decay of polarized τ lepton

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The polarization effects in the one-meson radiative decay of the polarized τ lepton, $\tau^- \rightarrow \pi^- \gamma \nu_\tau$, are investigated. The inner bremsstrahlung and structural amplitudes are taken into account. The asymmetry of the differential decay width caused by the τ lepton polarization and the Stokes parameters of the emitted photon itself, are calculated depending on the polarization of the decaying τ lepton. The numerical estimation of these physical quantities was done for arbitrary direction of the τ lepton polarization 3-vector in the rest frame. The vector and axial-vector form factors describing the structure-dependent part of the decay amplitude are determined using the chiral effective theory with resonances (R χ T).

1. INTRODUCTION

As it is known, the τ lepton is the only existing lepton which can decay, due to its large mass, into final states containing hadrons. The energy region of these decays corresponds to the hadron dynamics which is described by the non-perturbative QCD. Since the complete theory of non-perturbative QCD is absent at present, the phenomena in this energy region are described using various phenomenological approaches. To test different theoretical models it is important to investigate experimentally the hadronization processes of the weak currents. The semileptonic τ lepton decays are very suitable for such investigations since the leptonic weak interaction is well understood in the Standard Model (SM). A review of the present status of τ physics can be found in Ref. [1].

In the last decade the experimental investigations of the τ lepton decays are strongly

enlarged due to the construction of the B-factories with very high luminosity (BaBar, Belle) $L \approx 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$ [2]. At present the experiments at the B-factories led to the accumulation of the data sets of more than 10^9 τ lepton pairs [3]. Interesting results obtained at the B-factories revived the plans on the construction of the new facilities such as SuperKEKB (Japan) and Super $c - \tau$ (Russia) [2, 4, 5]. These projects will use a new technique to collide the electron-positron beams which permits to increase the existing luminosity by one or two orders of magnitude. The designed luminosity is $L \approx (1 - 2) \times 10^{35} \text{ cm}^{-2} \text{ s}^{-1}$ for the Super $c - \tau$ and $L \approx 10^{36} \text{ cm}^{-2} \text{ s}^{-1}$ for the SuperKEKB [2]. Besides, the Super $c - \tau$ and SuperKEKB factories can have a longitudinally polarized electron beam with the polarization degree of more than 80% and this guarantees production of polarized τ leptons.

This very high luminosity of the planned τ factories will allow one to perform the precise measurements of various decays of the τ lepton and thus permits to search for the manifestation of the new physics beyond the SM, such as a search for the lepton flavor violation, CP/T violation in the lepton sector and so on.

The simplest semileptonic τ lepton decay is $\tau^- \rightarrow \pi^-(K^-)\nu_\tau$, however in this case the hadronization of the weak currents is described by the form factors at fixed value of the momentum transfer squared t (t is the difference of the τ^- and ν_τ 4-momenta squared). The dependence of the form factors on this variable can be determined, in principle, in the transition $W \rightarrow \pi(K)\gamma$, where t is the squared invariant mass of the $\pi(K) - \gamma$ system. This transition can be investigated in the τ lepton radiative decay $\tau^- \rightarrow \pi^-(K^-)\nu_\tau\gamma$.

The one-pseudoscalar meson radiative τ lepton decays have been investigated in a number of papers [4–10]. The authors of Ref. [6] obtained the expression for the double-differential decay rate for the $\tau^- \rightarrow \nu_\tau\pi^-\gamma$ decay in terms of the vector $v(t)$ and axial-vector $a(t)$ form factors. The numerical estimates were done for the real parameterizations of these form factors using the vector-meson dominance approach. But the assumption that the form factors are real functions is not generally true since in the time-like region of the momentum transfer squared, which is the case for considered decay, these form factors are complex functions.

The author of Ref. [7] has studied the following radiative decays $\tau^- \rightarrow \nu_\tau\pi^-\gamma$ and $\tau^- \rightarrow \nu_\tau\rho^-\gamma$, obtained the analytical formulas for the differential decay rates and evaluated them assuming that the form factors are constant. The authors of Ref. [9] have studied the decays $\tau^- \rightarrow \nu_\tau\pi^-(K^-)\gamma$. They obtained the photon energy spectrum, the meson-photon

invariant mass distribution, and the integrated rates. The inner bremsstrahlung contribution to the decay rate contains infrared divergences and that is why the integrated decay rates must depend on the photon energy cut-off (or meson-photon invariant mass cut-off). For the photon energy cut-off 100 MeV, the integrated decay rates $R = \Gamma(\tau \rightarrow \nu_\tau \pi \gamma) / \Gamma(\tau \rightarrow \nu_\tau \pi) = 1.4 \cdot 10^{-2}$ [6] and $R = 1.0 \cdot 10^{-2}$ [9] were obtained. Note that the leptonic radiative decay of the τ lepton $\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau \gamma$ was measured with a branching ratio of $3.6 \cdot 10^{-3}$ [16]. So one can expect that the one-pseudoscalar meson radiative τ lepton decay $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ can also be measured experimentally since theory predicts for its branching ratio the value of the same order as for the $\tau^- \rightarrow \mu^- \bar{\nu}_\mu \nu_\tau \gamma$ decay.

Some polarization observables in the decay $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ have been considered in Ref. [10]. The general expressions for the Stokes parameters of the produced photon have been calculated. The influence of the possible anomalous magnetic moment of the τ lepton and existence of excited neutrinos on the matrix element of this decay are briefly discussed. The authors showed that a measurement of the dependence of the differential decay rate on the photon energy (at a fixed sum of the photon and pion energies) allows to determine the moduli and phases of the form factors as functions of the variable t .

The τ lepton radiative decays $\tau^- \rightarrow \nu_\tau \pi^- (K^-) \gamma$ were also studied in Refs. [11, 12] for the case of unpolarized particles. The light front quark model was used to evaluate the form factors $v(t)$ and $a(t)$ describing the structure-dependent contribution to these decays [11] and they found that in the SM the decay width is $\Gamma = 1.62 \cdot 10^{-2} (3.86 \cdot 10^{-3}) \Gamma(\tau \rightarrow \nu \pi)$ for the photon energy cut-off of 50 (400) MeV. The same decays were studied in Ref. [12]. The photon energy spectrum, pion-photon invariant mass distribution and integrated decay rate were calculated without free parameters and the authors obtained for the decay width values $1.46 \cdot 10^{-2} (2.7 \cdot 10^{-3}) \Gamma(\tau \rightarrow \nu \pi)$ for the same cut-off conditions.

The τ lepton decay for the case of the virtual photon which converted into the lepton-antilepton pair has been investigated in Refs. [13]. This decay was not measured up to now, but the cross channel (namely, $\pi^+ \rightarrow e^+ \nu_e e^+ e^-$) has been already measured [14, 15]. This decay and decay $\tau \rightarrow \pi l l \nu_\tau$ probe the transition $W^* \rightarrow \pi \gamma^*$ where both bosons (W and photon) are virtual. These decays complement the decay we consider in this paper and which can be a source of information about the transition $W^* \rightarrow \pi \gamma$. The vector and axial-vector form factors are the functions of two variables (instead of one as in our case) due to the virtuality of the photon and a third form factor appears in this case. The authors [13]

calculated the branching ratios and di-lepton invariant mass spectra. They predicted that the process with $l = e$ should be measured soon at B-factories.

Since the SuperKEKB and Super $c - \tau$ factories plan to have the longitudinally polarized electron beam with the polarization degree of the beam about 80%, it is worthwhile to investigate the polarization of the τ lepton. In this paper we investigated the polarization effects in the one-meson radiative decay of the τ lepton, $\tau^- \rightarrow \pi^- \gamma \nu_\tau$. The decay polarization asymmetry and the Stokes parameters of the emitted photon itself have been calculated for the case of the polarized τ lepton. Numerical estimates of these observables were done for arbitrary polarization of the τ lepton.

The vector and axial-vector form factors (which are of theoretical and experimental interest), describing the structure-dependent part of the decay amplitude, are determined in the framework of the chiral effective theory with resonances (R χ T) [17, 18]. The R χ T is an extension of the Chiral Perturbation Theory to the region of energies around 1 GeV, which includes explicitly the meson resonances. The corresponding Lagrangian contains a few free parameters, or coupling constants, and at the same time has a good predictive power. This approach has further theoretical developments, such as in Refs. [19, 20], and applications to various aspects of meson phenomenology (see, *e.g.*, review [21]). Here we would like to mention earlier studies of e^+e^- annihilation to the pair of pseudoscalar mesons with final-state radiation [22], radiative decays with light scalar mesons [23], two-photon form factors of the π^0 , η and η' mesons [24], and three-pion decay of the τ^- lepton [25].

The paper is organized as follows. In Sec. 2 the matrix element of the decay $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ is considered, and the phase space factor of the final particles is introduced for an unpolarized and polarized τ lepton, Stokes parameters and spin-correlation parameters of the photon are defined. It is done with help of the current tensor $T^{\mu\nu}$ and two unit space-like orthogonal 4-vectors which describe the polarization states of photon and which we express via particle 4-momenta. In Sec. 3 the current tensor is calculated in terms of the particle 4-momenta and the τ lepton polarization 4-vector. In Sec. 4 we describe the chosen model for the vector and axial-vector form factors which enter the structural part of the decay amplitude. In Sec. 5 results of some analytical and numerical calculations are presented and illustrated. Sec. 6 contains a discussion and conclusion. In Appendix A the formalism of R χ T is briefly reviewed. In Appendix B we consider a polarization of the τ^- lepton in the annihilation process $e^+e^- \rightarrow \tau^+\tau^-$ near threshold for a longitudinally polarized electron.

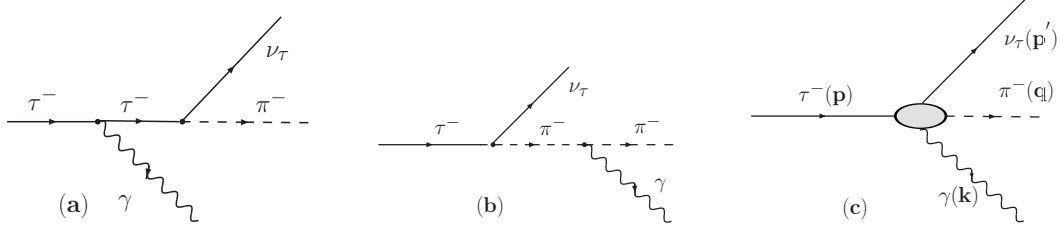


Figure 1. The Feynman diagrams for the radiative $\tau^- \rightarrow \pi^- + \nu_\tau + \gamma$ decay. The diagrams (a) and (b) correspond to the so-called structure-independent inner bremsstrahlung for which it is assumed that the pion is a point-like particle. The diagram (c) represents the contribution of the structure-dependent part and it is parameterized in terms of the vector and axial-vector form factors.

2. GENERAL FORMALISM

The main goal of our study is the investigation of polarization effects in the radiative semileptonic decay of a polarized τ lepton

$$\tau^-(p) \rightarrow \nu_\tau(p') + \pi^-(q) + \gamma(k). \quad (1)$$

2.1. Amplitude and decay width

The corresponding Feynman diagrams for the decay amplitude are shown in Fig. 1. The pole diagrams (a) and (b) describe the inner bremsstrahlung (IB) by the charged particles in the point-like approximation; diagram (c) describes the so-called structural radiation.

Thus, we have [9, 26]

$$M_\gamma = M_{IB} + M_S.$$

The IB piece, in the case of real photon ($k^2 = 0$), coincides with its so-called "contact limit" value and reads

$$iM_{IB} = ZM\bar{u}(p')(1 + \gamma_5) \left[\frac{\hat{k}\gamma^\mu}{2(kp)} + \frac{Ne_1^\mu}{(kp)(kq)} \right] u(p)\varepsilon_\mu^*(k), \quad (2)$$

where the dimensional factor Z incorporates all constants: $Z = eG_F V_{ud} F_\pi$, M is the τ lepton mass and $\varepsilon_\mu(k)$ is the photon polarization 4-vector. Here $e^2/4\pi = \alpha = 1/137$, $G_F = 1.166 \cdot 10^{-5} GeV^{-2}$ is the Fermi constant of the weak interactions, $V_{ud} = 0.9742$ is

the corresponding element of the CKM-matrix [27], $F_\pi = 92.42 \text{ MeV}$ is the constant which determines the decay $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$. The remaining notation is

$$e_1^\mu = \frac{1}{N} [(pk)q^\mu - (qk)p^\mu] , \quad (e_1 k) = 0 , \quad e_1^2 = -1 ,$$

$$N^2 = 2(qp)(pk)(qk) - M^2(qk)^2 - m^2(pk)^2 ,$$

where m is the pion mass.

The structure radiation in the decay (1) arises due to the possibility of the virtual radiative transition

$$W^- \rightarrow \pi^- + \gamma.$$

We write the corresponding amplitude in a standard form in terms of two complex form factors: vector $v(t)$ and axial-vector $a(t)$ [12, 26]

$$iM_R = \frac{Z}{M^2} \bar{u}(p') (1 + \gamma_5) \left\{ i\gamma_\alpha (\alpha\mu k q) v(t) - [\gamma^\mu (qk) - q^\mu \hat{k}] a(t) \right\} u(p) \varepsilon_\mu^*(k) , \quad (3)$$

where $t = (k + q)^2$, and

$$(\alpha\mu k q) = \epsilon^{\alpha\mu\nu\rho} k_\nu q_\rho , \quad \epsilon^{0123} = +1 , \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 , \quad \text{Tr} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\lambda = -4i\epsilon^{\mu\nu\rho\lambda} .$$

One can see that both matrix elements, M_{IB} and M_R , satisfy the condition of gauge invariance and for M_R it is valid for any choice of form factors.

The form factors play an important role in the low-energy hadron phenomenology. However, the experimental values of $v(0)$ and $a(0)$ have uncertainties both in absolute values and signs [16]. Of course, such a situation complicates a search for the signals of new physics beyond SM in future experiments with high statistic at τ -factories [28, 29].

To define the vector and axial-vector form factors we use the model based on the Resonance Chiral Theory [17]. The brief physical description of the theoretical approach to this problem is given in Appendix A and Section 4. In accordance with the results of the theoretical model used, we can write the form factors in the following form

$$a(t) = -f_A(t) \frac{M^2}{\sqrt{2}mF_\pi} , \quad v(t) = -f_V(t) \frac{M^2}{\sqrt{2}mF_\pi} ,$$

where $f_A(t)$ and $f_V(t)$ are defined in Section 4.

We use such a normalization that the differential width of the decay (1), in terms of the matrix element M_γ , has the following form in the τ lepton rest system

$$d\Gamma = \frac{1}{4M(2\pi)^5} |M_\gamma|^2 \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \delta(p'^2) , \quad (4)$$

where ω and ϵ are the energies of the photon and π meson. When writing $|M_\gamma|^2$ one has to use

$$u(p)\bar{u}(p) = (\hat{p} + M) , \quad u(p)\bar{u}(p) = (\hat{p} + M)(1 + \gamma_5 \hat{S})$$

for unpolarized and polarized τ lepton decays, respectively. Here S is the 4-vector of τ lepton polarization.

2.2. Phase space factor

It is convenient to analyze events of τ decay in its rest frame. In this system, in an unpolarized case, $|M_\gamma|^2$ depends on the photon and pion energies only: ω and ϵ . Then the phase space factor for unpolarized τ can be written as [10]

$$d\Phi = \frac{d^3k}{2\omega} \frac{d^3q}{2\epsilon} \delta(p'^2) = \pi^2 d\omega d\epsilon , \quad (5)$$

and the region of variation of the energies is defined by the following inequalities

$$\begin{aligned} \frac{M^2 + m^2 - 2M\epsilon}{2(M - \epsilon + |\mathbf{q}|)} \leq \omega \leq \frac{M^2 + m^2 - 2M\epsilon}{2(M - \epsilon - |\mathbf{q}|)} , \quad m \leq \epsilon \leq \frac{M^2 + m^2}{2M} , \\ \frac{M^2 + m^2 + 4\omega(\omega - M)}{2(M - 2\omega)} \leq \epsilon \leq \frac{M^2 + m^2}{2M} , \quad 0 \leq \omega \leq \frac{M^2 - m^2}{2M} . \end{aligned} \quad (6)$$

As the τ radiative decay amplitude depends on the invariant variable $t = (k + q)^2 = M(2\epsilon + 2\omega - M)$ via the vector and axial-vector form factors in the amplitude (3), one can perform integration of the differential width with respect to ϵ (or ω) at fixed values of t to investigate these form factors. It may be done noting that

$$d\epsilon d\omega = \frac{1}{2M} d\epsilon dt = \frac{1}{2M} d\omega dt$$

and

$$\frac{t^2 + m^2 M^2}{2Mt} \leq \epsilon \leq \frac{M^2 + m^2}{2M} ; \quad \frac{t - m^2}{2M} \leq \omega \leq \frac{M(t - m^2)}{2t} ; \quad m^2 \leq t \leq M^2 . \quad (7)$$

The integration regions for the variables (ϵ, ω) , (ϵ, t) and (ω, t) are shown in Fig. 2.

In a polarized case we have an additional independent 4-vector S . In the τ lepton rest frame $S = (0, \mathbf{n})$. We define a coordinate system with an axis OZ directed along the vector \mathbf{n} and the pion 3-momentum lies in the plane XZ, as it is shown in Fig. 3. If one used the

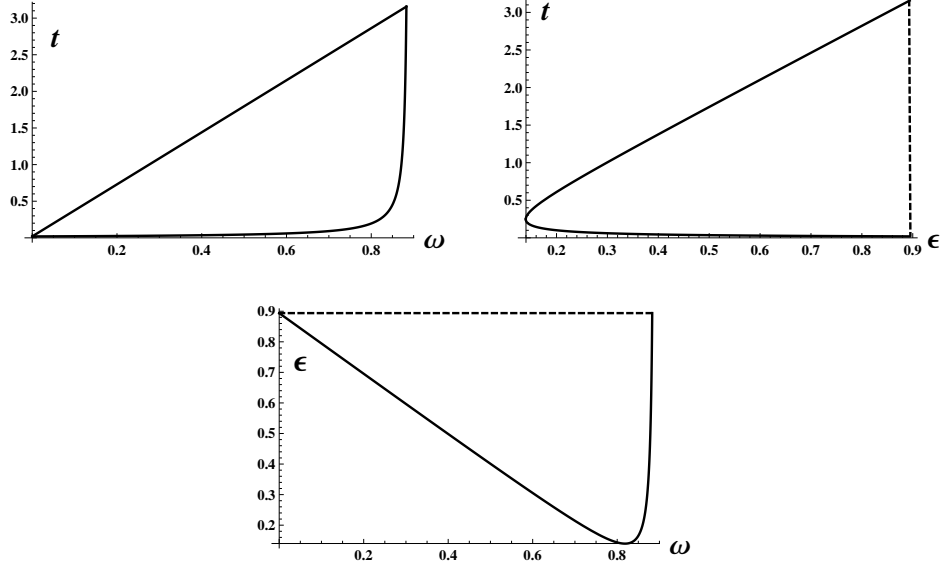


Figure 2. The regions of variation for the radiative τ^- decay for different sets of kinematical variables. The energies of the photon and pion (ω , ϵ) are given in GeV and the invariant variable t in GeV^2 . The line equations are defined by the inequalities (6) and (7).

δ - function $\delta(p'^2)$ to eliminate integration over the azimuthal angle φ , the phase space (5) can be rewritten in the form

$$d\Phi = \frac{\pi dc_1 dc_2 d\omega d\epsilon}{K}, \quad K = s_1 s_2 |\sin \varphi| \sqrt{1 - c_1^2 - c_2^2 - c_{12}^2 + 2c_1 c_2 c_{12}}, \quad (8)$$

where $c_1(s_1) = \cos \theta_1 (\sin \theta_1)$, $c_2(s_2) = \cos \theta_2 (\sin \theta_2)$, $c_{12} = \cos \theta_{12}$, and $\theta_1, \theta_2, \theta_{12}$ are the angles between \mathbf{n} and \mathbf{q} , \mathbf{n} and \mathbf{k} , \mathbf{q} and \mathbf{k} , respectively. In this case we can study the spin-dependent effects caused only by the terms, in the matrix element squared, which do not depend on the azimuthal angle, namely (Sk) and (Sq) . The contribution of the term containing $(Spqk)$ vanishes when we integrate over φ in the whole region $(0, 2\pi)$.

The limits of variation of c_1 and c_2 are defined by the condition of positiveness of the expression under the square root in Eq. (8) and they are shown in Fig. 3. A simple calculation gives

$$c_{1-} \leq c_1 \leq c_{1+}, \quad -1 \leq c_2 \leq 1, \quad c_{1\pm} = c_2 c_{12} \pm s_2 s_{12},$$

where $s_{12} = \sin \theta_{12}$. Besides, the integration in the above limits gives

$$\int \frac{dc_1}{K} = \int \frac{dc_2}{K} = \pi, \quad \int \frac{c_1 dc_1}{K} = \pi c_2 c_{12}, \quad \int \frac{c_2 dc_2}{K} = \pi c_1 c_{12}. \quad (9)$$

If $|M_\gamma|^2$ has no angular dependence, it corresponds to the unpolarized case.

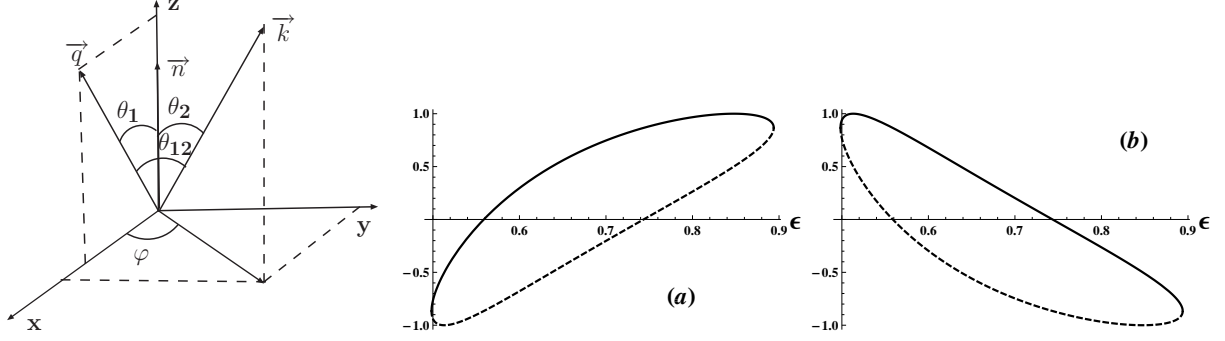


Figure 3. Definition of the angles for the case of the polarized radiative τ decay at rest (left picture) ; the limits of variation of c_1 at $\omega = 0.4$ GeV and $c_2 = 150^\circ$ (a) and $c_2 = 30^\circ$ (b): the solid (dashed) line corresponds to c_{1+} (c_{1-}).

2.3. Decay asymmetry and Stokes parameters

To determine moduli and phases of the form factors it is necessary to measure some polarization observables. The simplest of such observables are the so-called single-spin quantities: the asymmetries caused by the τ lepton polarization and the photon Stokes parameters. We consider also the double-spin quantities: the dependence of the Stokes parameters on the τ lepton polarization.

Generally, the photon polarization properties are described by its Stokes parameters. At this point we have to clarify the terminology used. The measurable Stokes parameters $\bar{\xi}_i, i = 1, 2, 3$ define the covariant spin-density matrix of a photon in terms of quantities $\bar{\xi}_i$ and two independent polarization 4-vectors $e_{1\mu}$ and $e_{2\mu}$ [30]

$$\rho_{\mu\nu} = \frac{1}{2} [e_{1\mu}e_{1\nu} + e_{2\mu}e_{2\nu} + \bar{\xi}_1(e_{1\mu}e_{2\nu} + e_{1\nu}e_{2\mu}) - i\bar{\xi}_2(e_{1\mu}e_{2\nu} - e_{1\nu}e_{2\mu}) + \bar{\xi}_3(e_{1\mu}e_{1\nu} - e_{2\mu}e_{2\nu})], \quad (10)$$

$$e_1^2 = e_2^2 = -1, \quad (ke_1) = (ke_2) = 0.$$

If the parameters $\bar{\xi}_i$ are measured, the matrix element squared can be written as contraction of the current tensor $T^{\mu\nu}$ and matrix $\rho_{\mu\nu}$,

$$|M_\gamma|^2 = T^{\mu\nu} \rho_{\mu\nu}, \quad (11)$$

where the current tensor $T^{\mu\nu}$ obeys the evident conditions due to the electromagnetic current conservation

$$k_\mu T^{\mu\nu} = T^{\mu\nu} k_\nu = 0.$$

The polarization states of a real photon are described by two independent pure space polarization vectors \mathbf{l}_1 and \mathbf{l}_2 , which are both perpendicular to its 3-momentum \mathbf{k} . In our case, in the rest system of the decaying τ^- lepton, it is convenient to take \mathbf{l}_1 in the decay plane and \mathbf{l}_2 to be perpendicular to this plane. One can determine two covariant polarization 4-vectors, which in the rest frame coincide with \mathbf{l}_1 and \mathbf{l}_2 . These vectors are defined as follows

$$\varepsilon_1^\mu = e_1^\mu - \frac{1}{N} \left[(qp) - M^2 \frac{(qk)}{(pk)} \right] k^\mu, \quad \varepsilon_2^\mu \equiv e_2^\mu = \frac{(\mu p q k)}{N}, \quad (12)$$

where e_1^μ and N are defined after Eq. (2).

It is easy to verify that in the rest frame

$$\varepsilon_{1\mu} = (0, \mathbf{l}_1), \quad \varepsilon_{2\mu} = (0, \mathbf{l}_2), \quad \mathbf{l}_1^2 = \mathbf{l}_2^2 = 1, \\ \mathbf{l}_1 = \frac{\mathbf{q} - \hat{\mathbf{k}}(\hat{\mathbf{k}}\mathbf{q})}{\sqrt{\mathbf{q}^2 - (\hat{\mathbf{k}}\mathbf{q})^2}}, \quad \mathbf{l}_2 = [\hat{\mathbf{k}} \mathbf{l}_1], \quad \hat{\mathbf{k}} = \frac{\mathbf{k}}{\omega}, \quad (\mathbf{l}_1 \hat{\mathbf{k}}) = (\mathbf{l}_2 \hat{\mathbf{k}}) = 0.$$

Therefore, the set of the unit 3-vectors \mathbf{l}_1 , \mathbf{l}_2 , and $\hat{\mathbf{k}}$ forms the right system of rectangular coordinates. Due to the electromagnetic current conservation, we can replace $\varepsilon_{1\mu}$ by $e_{1\mu}$ in calculations of any observable. Thus, in Eq. (10) we will use polarization 4-vectors in form (12). Then the matrix element squared reads

$$|M_\gamma|^2 = \frac{1}{2} [\Sigma + \Sigma_i \bar{\xi}_i], \quad (13)$$

where

$$\Sigma = T^{\mu\nu} (e_{1\mu} e_{1\nu} + e_{2\mu} e_{2\nu}), \quad \Sigma_1 = T^{\mu\nu} (e_{1\mu} e_{2\nu} + e_{1\nu} e_{2\mu}), \\ \Sigma_2 = -i T^{\mu\nu} (e_{1\mu} e_{2\nu} - e_{1\nu} e_{2\mu}), \quad \Sigma_3 = T^{\mu\nu} (e_{1\mu} e_{1\nu} - e_{2\mu} e_{2\nu}).$$

It is obvious that parameters ξ'_i depend on the properties of detectors which analyze the polarization states of the photon and do not depend on the production mechanism. On the contrary, quantities Σ and Σ_i are defined only by the decay amplitude and thus they determine polarization properties of the photon itself in the decay (1) [30]. To study predictions of different theoretical models for these quantities we can write

$$|M_\gamma|^2 = \Sigma + \Sigma_i$$

instead of the expression (13).

For a polarized τ lepton the current tensor reads

$$T_{\mu\nu} = T_{\mu\nu}^0 + T_{\mu\nu}^S,$$

where $T_{\mu\nu}^S$ depends on the polarization 4-vector of τ lepton and $T_{\mu\nu}^0$ does not. In this case we can write

$$\Sigma = \Sigma^0 + \Sigma^S, \quad \Sigma_i = \Sigma_i^0 + \Sigma_i^S,$$

and define the physical quantities

$$A^S = \frac{\Sigma^S d\Phi}{\Sigma^0 d\Phi}, \quad \xi_i = \frac{\Sigma_i^0 d\Phi}{\Sigma^0 d\Phi}, \quad \xi_i^S = \frac{\Sigma_i^S d\Phi}{\Sigma^0 d\Phi}, \quad (14)$$

which completely describe the polarization effects in considered decay.

The quantity A^S represents the polarization asymmetry of the differential decay width caused by the polarization of τ lepton. The quantities ξ_i are the Stokes parameters of the photon itself provided that τ lepton is unpolarized and the quantities ξ_i^S are the correlation parameters describing influence of τ lepton polarization on the photon Stokes parameters.

Thus, to analyze the polarization phenomena in the process (1) we have to study both the spin-independent and spin-dependent parts of the differential width and, in accordance with Eq. (4), they are

$$\frac{d\Gamma_0}{d\Phi} = g\Sigma^0, \quad \frac{d\Gamma_0^S}{d\Phi} = g\Sigma^S, \quad \frac{d\Gamma_i}{d\Phi} = g\Sigma_i^0, \quad \frac{d\Gamma_i^S}{d\Phi} = g\Sigma_i^S, \quad g = \frac{1}{4M(2\pi)^5}.$$

Note that, by partial integration in numerators and denominator in the relations (14), we can define and study also the corresponding reduced polarization parameters.

If we record the photon and pion energies, the parameters $\xi_1(\omega, \epsilon)$ and $\xi_3(\omega, \epsilon)$ describe linear polarizations of the photon and the parameter $\xi_2(\omega, \epsilon)$ - the circular one. The last parameter does not depend on the choice of two polarization vectors which, in our work, are defined by the relations (12). On the contrary, each of the parameters $\xi_1(\omega, \epsilon)$ and $\xi_3(\omega, \epsilon)$ depends on the axes relative to which it is defined, and only the quantity $\sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}$ remains invariant. In principle, one can choose the polarization vectors e'_1 and e'_2 so that to vanish, for example, $\xi'_1(\omega, \epsilon)$; then $\xi'_3(\omega, \epsilon) = \sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}$ (and vice versa).

This statement can be verified by a simple rotation of the 4-vectors e_1 and e_2 in the plane perpendicular to the direction \mathbf{k} [30]

$$e'_{1\mu} = e_{1\mu} \cos \beta - e_{2\mu} \sin \beta, \quad e'_{2\mu} = e_{1\mu} \sin \beta + e_{2\mu} \cos \beta \quad (15)$$

so that in terms of e'_1 and e'_2 we have, using definitions (13) and (14)

$$\xi'_1(\omega, \epsilon) = \xi_1(\omega, \epsilon) \cos 2\beta + \xi_3(\omega, \epsilon) \sin 2\beta,$$

$$\xi_3'(\omega, \epsilon) = -\xi_1(\omega, \epsilon) \sin 2\beta + \xi_3(\omega, \epsilon) \cos 2\beta, \quad \xi_2'(\omega, \epsilon) = \xi_2(\omega, \epsilon).$$

Taking, for example,

$$\cos 2\beta = \frac{\xi_1(\omega, \epsilon)}{\sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}}, \quad \sin 2\beta = \frac{\xi_3(\omega, \epsilon)}{\sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}},$$

one easily obtains

$$\xi_1'(\omega, \epsilon) = \sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}, \quad \xi_3'(\omega, \epsilon) = 0.$$

If the experimental setup allows to fix the decay plane, the Stokes parameter ξ_3 defines the probability of the photon linear polarization along two orthogonal directions: along \mathbf{l}_1 and \mathbf{l}_2 . If $\xi_3 = 1$, ($\xi_3 = -1$) the photon is fully polarized along the direction \mathbf{l}_1 , (\mathbf{l}_2) and its polarization vector lies in the decay plane (perpendicular to it). In general, the probability of the linear polarization in the decay plane is $(1 + \xi_3)/2$ and in the plane perpendicular to it is $(1 - \xi_3)/2$.

The parameter ξ_1 defines the probability of the linear polarization in the planes rotated by the angle $\phi = \pm 45^\circ$ around the \mathbf{k} -direction relative to the decay one. The full linear polarization at $\phi = +45^\circ$ (-45°) occurs at $\xi_1 = 1$ (-1). The corresponding probabilities, in a general case, are $(1 + \xi_1)/2$ and $(1 - \xi_1)/2$, respectively. Thus, we can say that the circular polarization degree of a photon equals $\xi_2(\omega, \epsilon)$ ($\xi_2(\omega, \epsilon) = 1$ (-1) corresponds to full right (left) circular polarization) and linear one equals $\sqrt{\xi_1(\omega, \epsilon)^2 + \xi_3(\omega, \epsilon)^2}$. Nevertheless, both parameters $\xi_1(\omega, \epsilon)$ and $\xi_3(\omega, \epsilon)$ are measurable and carry different information about the decay mechanism. But to define them separately one has to determine the plane (\mathbf{q}, \mathbf{k}) in every event. The same regards also reduced photon polarization parameters, for example, $\xi_i(\omega)$ and so on.

2.4. Polarization of τ lepton

Before proceeding further, let us discuss briefly the possible polarization states of a τ lepton created at τ -factories in electron-positron annihilation process with a longitudinally polarized electron beam

$$e^- + e^+ \rightarrow \tau^- + \tau^+.$$

Simple calculations in the lowest approximation of QED show that polarization of τ arises if at least one of the colliding beams is polarized. For example, in the case of a longitudinally

polarized electron beam, the τ^- lepton has longitudinal and transverse polarizations in the reaction plane relative to the τ lepton momentum direction (for the details see Appendix B)

$$P^L = \frac{2\lambda \cos \theta}{Q}, \quad P^T = \frac{4\lambda M \sin \theta}{\sqrt{s}Q}, \quad Q = 1 + \cos^2 \theta + \frac{4M^2}{s} \sin^2 \theta, \quad (16)$$

where λ is the beam polarization degree, θ is the angle between 3-momenta of the electron and τ^- lepton in c.m.s. and s is the total energy squared in c.m.s. If we select and analyze the events with a longitudinally (transversally) polarized τ^- , then 3-vector \mathbf{n} in the rest system (see Fig. 3) lies in the annihilation reaction plane and it is directed along (perpendicular to) the 3-momentum of τ^- in c.m.s.

3. CALCULATION OF THE CURRENT TENSOR $T^{\mu\nu}$

The current tensor $T^{\mu\nu}$ contains three contributions – the IB, the resonance ones and the interference between the IB and resonance amplitudes. We divide every contribution into the symmetric and antisymmetric parts relative to the Lorentz indices. The symmetric part contributes to Σ , Σ_1 and Σ_3 whereas the antisymmetric one – to Σ_2 only.

Below, in the formulas for the different parts of the current tensor we omitted the terms proportional to the 4-vectors k_μ and k_ν since they do not contribute to the observables.

3.1. Inner bremsstrahlung contribution

For the IB contribution we have

$$T_{IB}^{\mu\nu} = 4Z^2 M^2 [S_{IB}^{\mu\nu} + iA_{IB}^{\mu\nu}],$$

where the symmetric part reads

$$S_{IB}^{\mu\nu} = \frac{(qk) - (pk)}{(pk)^2} [(pk) + M(kS)] g^{\mu\nu} + \frac{N^2}{(pk)^2 (qk)^2} [M^2 - m^2 + 2M(p'S)] e_1^\mu e_1^\nu + \quad (17)$$

$$\frac{NM}{(pk)^2 (qk)} [(e_1 l_p)^{\mu\nu} - (e_1 l_q)^{\mu\nu}],$$

and antisymmetric one is

$$A_{IB}^{\mu\nu} = \frac{(pk) - (qk)}{(pk)^2} [M(\mu\nu kS) - (\mu\nu pk)] + \frac{N^2}{(pk)^2 (qk)} [e_1 e_2]^{\mu\nu} - \quad (18)$$

$$\frac{MN}{(pk)^2(qk)}[e_1^\mu(\nu p' k S) - e_1^\nu(\mu p' k S)] .$$

Here we used the following notation

$$l_p^\mu = (pk)S^\mu - (kS)p^\mu , \quad l_q^\mu = (qk)S^\mu - (kS)q^\mu ,$$

$$(ab)^{\mu\nu} = a^\mu b^\nu + a^\nu b^\mu , \quad [ab]^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu , \quad (kl_p) = (kl_q) = 0 .$$

Note that the form of the antisymmetric part can be written in different equivalent forms. Indeed, one can derive another form using the well known relation

$$g^{\alpha\beta}(\mu\nu\lambda\rho) = g^{\alpha\mu}(\beta\nu\lambda\rho) + g^{\alpha\nu}(\mu\beta\lambda\rho) + g^{\alpha\lambda}(\mu\nu\beta\rho) + g^{\alpha\rho}(\mu\nu\lambda\beta) .$$

3.2. Resonance contribution

As concern the resonance contribution into the current tensor $T^{\mu\nu}$, we write it in the form

$$T_R^{\mu\nu} = \frac{8Z^2}{M^4} [S_R^{\mu\nu} + iA_R^{\mu\nu}] ,$$

where both the symmetric and antisymmetric parts include four independent pieces. They are proportional to $|a(t)|^2$, $|v(t)|^2$, $Re(a(t)v^*(t))$ and $Im(a(t)v^*(t))$. Denoting the respective symmetrical pieces as S_{Ra} , S_{Rv} , S_{Rr} and S_{Ri} and the antisymmetrical ones as A_{Ra} , A_{Rv} , A_{Rr} and A_{Ri} , we have

$$S_{Ra}^{\mu\nu} = |a(t)|^2 \{ (qk)^2 [M(p'S) - (pp')] g^{\mu\nu} + 2N^2 e_1^\mu e_1^\nu + NM(e_1 l_q)^{\mu\nu} \} , \quad (19)$$

$$A_{Ra}^{\mu\nu} = |a(t)|^2 (qk) \{ (qp')(\mu\nu pk) - (qp)(\mu\nu p'k) - M[(qp')(\mu\nu Sk) - (qS)(\mu\nu p'k)] \} . \quad (20)$$

$$S_{Rv}^{\mu\nu} = |v(t)|^2 \{ [M(p'S) - (pp')] (qk)^2 g_{\mu\nu} + 2N^2 e_2^\mu e_2^\nu - NM[e_2^\mu(\nu qkS) + e_2^\nu(\mu qkS)] \} , \quad (21)$$

$$A_{Rv}^{\mu\nu} = |v(t)|^2 (\mu\nu qk) \{ (qk)[(pq) - (pk)] - m^2(pk) - M[(qp')(kS) - (p'k)(qS)] \} , \quad (22)$$

$$S_{Rr}^{\mu\nu} = 2Re\{a^*(t)v(t)\} g^{\mu\nu} (qk) \{ (qk)[(pk) - (pq)] + m^2(pk) \} \quad (23)$$

$$- M[(kp')(Sq) - (Sk)(p'q)] \} ,$$

$$A_{Rr}^{\mu\nu} = 2Re\{a^*(t)v(t)\} \{ (qk)[(pp') - M(p'S)](\mu\nu qk) - N^2[e_1 e_2]^{\mu\nu} + \quad (24)$$

$$\frac{1}{2} NM(e_1^\mu(\nu qkS) - e_1^\nu(\mu qkS) + [e_2 l_q]^{\mu\nu}) \} ,$$

$$S_{Ri}^{\mu\nu} = 2Im\{a^*(t)v(t)\} \{ \frac{1}{2} NM[-(e_2 l_q)^{\mu\nu} + e_1^\mu(\nu qkS) + e_1^\nu(\mu qkS)] - N^2(e_1 e_2)^{\mu\nu} \} , \quad (25)$$

$$A_{Ri}^{\mu\nu} = 0 . \quad (26)$$

The interference between the vector and axial-vector contributions of the resonance amplitudes is sensitive to the relative sign of the axial-vector and vector couplings and its separation can be used to fix the sign of the ratio $f_V(0)/f_A(0)$.

3.3. IB – Resonance interference

The interference between the IB and resonance amplitudes is more sensitive to all the resonance parameters because $a(t)$ and $v(t)$ enter it linearly. It is very important to find such a polarization observable where the interference contribution would be enhanced relative to the background created by pure IB and resonance contributions. For the current tensor caused by the IB - resonance interference we define

$$T_{IR}^{\mu\nu} = \frac{8Z^2}{M(pk)} [S_{IR}^{\mu\nu} + iA_{IR}^{\mu\nu}] .$$

Again we have four symmetric and antisymmetric terms, which read

$$S_{IRra}^{\mu\nu} = Re(a(t)) \left\{ (qk)[(pk)(Sp') - (kS)(pp') - M(p'k)]g^{\mu\nu} + \frac{2N^2}{(qk)}[M + (Sp')]e_1^\mu e_1^\nu + \right. \quad (27)$$

$$\left. \frac{N(pp')}{(qk)}(e_1 l_q)^{\mu\nu} + N(e_1 l_p)^{\mu\nu} \right\} ,$$

$$A_{IRra}^{\mu\nu} = Re(a(t)) \left\{ [(qS)(2(pk) - (qk) + (pq)) - M(qk)](\mu\nu pk) + \right. \quad (28)$$

$$\left. [(pq)(2(pk) - (qk) + (pq)) - M^2(m^2 + (qk))](\mu\nu kS) + M[(qk) - M(qS)](\mu\nu qk) \right\} ,$$

$$S_{IRia}^{\mu\nu} = -Im(a(t))N \left[(qk)(e_2 S)g^{\mu\nu} + \frac{(e_1 Q)^{\mu\nu}}{(qk)} \right] , \quad (29)$$

$$A_{IRia}^{\mu\nu} = Im(a(t))N \left[[e_1 l_p]^{\mu\nu} - \left(1 + \frac{(pp')}{(qk)} \right) [e_1 l_q]^{\mu\nu} \right] , \quad (30)$$

where

$$Q^\mu = N(e_2 S)q^\mu - (qk)(Spq\mu) ,$$

.

$$S_{IRrv}^{\mu\nu} = Re(v(t)) \left\{ (p'k)[M(qk) + (pq)(kS) - (pk)(qS)]g^{\mu\nu} + 2\frac{N^2}{(qk)}(qS)e_1^\mu e_1^\nu + \right. \quad (31)$$

$$\left. N \left[\frac{(pq)}{(qk)}(e_1 l_q)^{\mu\nu} - \left(\frac{1}{2} + \frac{m^2}{(qk)} \right) (e_1 l_p)^{\mu\nu} + \frac{1}{2}(e_2^\mu(\nu pkS) + e_2^\nu(\mu pkS)) \right] \right\} ,$$

$$A_{IRrv}^{\mu\nu} = Re(v(t)) \left\{ [M(p'k) + (pp')(kS) - (pk)(p'S)](\mu\nu qk) - \frac{N^2}{(qk)} [M + (p'S)][e_1 e_2]^{\mu\nu} + \right. \quad (32)$$

$$\left. N \left[\frac{(pp')}{(qk)} [e_2 l_q]^{\mu\nu} + [e_2 l_p]^{\mu\nu} \right] \right\} ,$$

$$S_{IRiv}^{\mu\nu} = Im(v(t)) N \left\{ -\frac{N}{(qk)} [M + (p'S)](e_1 e_2)^{\mu\nu} + \left(\frac{1}{2} + \frac{(pp')}{(qk)} \right) (e_1^\mu (\nu qkS) + e_1^\nu (\mu qkS)) - \right. \quad (33)$$

$$\left. \frac{1}{2} (e_2 (2l_p - l_q))^{\mu\nu} \right\} ,$$

$$A_{IRiv}^{\mu\nu} = Im(v(t)) N \left\{ \left(\frac{1}{2} + \frac{m^2}{(qk)} \right) [e_1 l_p]^{\mu\nu} - \frac{(pq)}{(qk)} [e_1 l_q]^{\mu\nu} + \right. \quad (34)$$

$$\left. \frac{1}{2} [e_2^\mu (\nu pkS) - e_2^\nu (\mu pkS)] - [e_2^\mu (\nu qkS) - e_2^\nu (\mu qkS)] \right\} .$$

In the above formulas we omitted terms proportional to the 4-vectors k_μ and k_ν since they do not contribute to any observables.

The expression for the current tensor $T^{\mu\nu}$ allows us to derive all the polarization observables in the Lorentz invariant form by contracting this tensor with an appropriate combination of 4-vectors e_1 and e_2 . The set of needed formulas reads

$$\begin{aligned} e_1^2 &= e_2^2 = -1 , \quad (e_1 e_2) = 0 , \\ (e_1 l_p) &= -(e_2 pkS) = \frac{1}{N} \left\{ (Sq)(pk)^2 + (Sk)[M^2(qk) - (pq)(pk)] \right\} , \\ (e_2 l_p) &= (e_1 pkS) = \frac{(pk)}{N} (Spqk) , \\ (e_1 l_q) &= -(e_2 qkS) = \frac{1}{N} \left\{ (Sq)(pk)(qk) + (Sk)[(pq)(qk) - m^2(pk)] \right\} , \\ (e_2 l_q) &= (e_1 qkS) = \frac{(qk)}{N} (Spqk) , \\ (e_1 e_2 qk) &= -(qk) , \quad (e_1 e_2 pk) = -(pk) , \quad (e_1 e_2 Sk) = -(Sk) , \\ (e_1 Q) &= \frac{[m^2(pk) - (pq)(qk)]}{N} (Spqk) , \\ (e_2 Q) &= \frac{(qk)}{N} \left\{ (kS)[(pq)^2 - M^2 m^2] + (qS)[M^2(qk) - (pq)(pk)] \right\} . \end{aligned} \quad (35)$$

In the presence of a polarized τ lepton the structure of the differential width and the Stokes parameters of the photon are much richer. As we saw below, in the polarized electron-positron annihilation the created τ leptons have essential longitudinal or transverse polarizations. In both cases, in the rest system of τ , its polarization 4-vector $S^\mu = (0, \mathbf{n})$, and choosing a coordinate system as shown in Fig. 3 one has

$$(qS) = -|\mathbf{q}|c_1, \quad (kS) = -\omega c_2, \quad (36)$$

$$(Spqk) = M(\mathbf{n}[\mathbf{q}\mathbf{k}]) = \text{sign}(\sin \varphi) M|\mathbf{q}|\omega \sqrt{1 - c_1^2 - c_2^2 - c_{12}^2 + 2c_1c_2c_{12}},$$

where $\sin \varphi$ defines the y -component of the 3-vector $|\mathbf{k}|$, namely $k_y = \omega \sin \theta_2 \sin \varphi$.

If one sums events with all possible values of the azimuthal angle φ , the spin correlation $(Spqk)$, which is perpendicular to the plane (\mathbf{q}, \mathbf{k}) , does not contribute. On the other hand, spin correlations in the plane (\mathbf{q}, \mathbf{k}) , caused by the (Sq) and (Sk) terms, being integrated over c_1 (or over c_2), are always proportional to c_2 (or c_1) as it follows from the relations (9) and (36).

4. AXIAL-VECTOR AND VECTOR FORM FACTORS IN THE $R_{\chi T}$

From the Lagrangians in Appendix A one can obtain the photon interaction with the pseudoscalar mesons P in the following form

$$\begin{aligned} \mathcal{L}_{\gamma PP} &= ieB^\mu \text{Tr} \left(Q, [\Phi, \partial_\mu \Phi] \right) = ieB^\mu (\pi^+ \overset{\leftrightarrow}{\partial}_\mu \pi^- + K^+ \overset{\leftrightarrow}{\partial}_\mu K^-), \\ \overset{\leftrightarrow}{\partial}_\mu &\equiv \overset{\rightarrow}{\partial}_\mu - \overset{\leftarrow}{\partial}_\mu. \end{aligned} \quad (37)$$

The axial transition $W^\pm \rightarrow \pi^\pm (K^\pm)$ is described by

$$\begin{aligned} \mathcal{L}_{WP} &= -\frac{gF}{2} \text{Tr} \left(W_\mu \partial^\mu \Phi \right) \\ &= \frac{gF}{2} W_\mu^+ (V_{ud} \partial^\mu \pi^- + V_{us} \partial^\mu K^-) + \text{h.c.} \end{aligned} \quad (38)$$

The vertex with an additional photon, $W^\pm \rightarrow \gamma \pi^\pm (K^\pm)$, is generated from

$$\begin{aligned} \mathcal{L}_{WP\gamma} &= -\frac{i}{2} egF B^\mu \text{Tr} \left([Q, \Phi] W_\mu \right) \\ &= \frac{i}{2} egF B^\mu W_\mu^+ (V_{ud} \pi^- + V_{us} K^-) + \text{h.c.} \end{aligned} \quad (39)$$

To evaluate the resonance contribution to the axial-vector form factor one needs the vertex of the $W^\pm \rightarrow a_1^\pm (K_1^\pm)$ transition

$$\begin{aligned}\mathcal{L}_{WA} &= -\frac{1}{4}gF_A \text{Tr}(W_{\mu\nu}A^{\mu\nu}) \\ &= -\frac{gF_A}{2}\partial_\mu W_\nu^+ (V_{ud}a_1^{-\mu\nu} + V_{us}K_1^{-\mu\nu}) + \text{h.c.}\end{aligned}\quad (40)$$

The transition $a_1^\pm (K_1^\pm) \rightarrow \gamma\pi^\pm (K^\pm)$ is described by

$$\mathcal{L}_{AP\gamma} = -i\frac{eF_A}{2F}F^{\mu\nu}(\pi^-a_{1\mu\nu}^+ + K^-K_{1\mu\nu}^+) + \text{h.c.}\quad (41)$$

The transition $W^\pm \rightarrow \pi^\pm\rho^0$ is generated from the Lagrangian

$$\mathcal{L}_{WPV} = -i\frac{gG_V}{\sqrt{2}F}\text{Tr}\left(V^{\mu\nu}[W_\mu, \partial_\nu\Phi]\right) + i\frac{gF_V}{4\sqrt{2}F}\text{Tr}\left(V^{\mu\nu}[W_{\mu\nu}, \Phi]\right)\quad (42)$$

with the notation

$$W_{\mu\nu} = W_{\mu\nu}^+T_+ + W_{\mu\nu}^-T_-, \quad W_{\mu\nu}^\pm \equiv \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm.\quad (43)$$

Keeping in Eq. (42) the contribution from the neutral vector mesons, one obtains

$$\begin{aligned}\mathcal{L}_{WPV} &= i\frac{gG_V}{\sqrt{2}F}W_\mu^+[-\sqrt{2}V_{ud}\partial_\nu\pi^-\rho^{0\mu\nu} + V_{us}\partial_\nu K^-(\phi - \frac{1}{\sqrt{2}}\rho^0 - \frac{1}{\sqrt{2}}\omega)^{\mu\nu}] \\ &\quad -i\frac{gF_V}{4\sqrt{2}F}W_{\mu\nu}^+[-\sqrt{2}V_{ud}\pi^-\rho^{0\mu\nu} + V_{us}K^-(\phi - \frac{1}{\sqrt{2}}\rho^0 - \frac{1}{\sqrt{2}}\omega)^{\mu\nu}] + \text{h. c.}\end{aligned}\quad (44)$$

Finally, one needs the term describing a transition of the neutral vector mesons to photon

$$\mathcal{L}_{V\gamma} = -i\frac{eF_V}{\sqrt{2}F}F^{\mu\nu}\text{Tr}\left(V_{\mu\nu}Q\right) = eF_VF^{\mu\nu}\left(\frac{1}{2}\rho_{\mu\nu}^0 + \frac{1}{6}\omega_{\mu\nu} - \frac{1}{3\sqrt{2}}\phi_{\mu\nu}\right)\quad (45)$$

with $F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$.

Collecting vertices from (40), (41) for the transition $W^- \rightarrow a_1^- \rightarrow \pi^-\gamma$, and vertices from (44), (45) for the transition $W^- \rightarrow \pi^-\rho^0 \rightarrow \pi^-\gamma$, we obtain the axial-vector form factor in the form

$$f_A(t) = \frac{\sqrt{2}m_{\pi^\pm}}{F_\pi}\left[\frac{F_A^2}{m_a^2 - t - im_a\Gamma_a(t)} + \frac{F_V(2G_V - F_V)}{m_\rho^2}\right],\quad (46)$$

where $\Gamma_a(t)$ is the decay width of the a_1 -meson.

Note that the axial-vector form factor in Eq. (46) is normalized at $t = 0$ as

$$f_A(0) = \frac{\sqrt{2}m_{\pi^\pm}}{F_\pi}\left[\frac{F_A^2}{m_a^2} + \frac{F_V(2G_V - F_V)}{m_\rho^2}\right],\quad (47)$$

which is consistent with the chiral expansion in order $\mathcal{O}(p^4)$ (see also Refs. [37, 38]) in terms of the low-energy constants,

$$\begin{aligned} f_A(0) &= \frac{4\sqrt{2}m_{\pi^\pm}}{F_\pi}(L_9 + L_{10}), \\ L_9 &= \frac{F_V G_V}{2m_\rho^2}, \quad L_{10} = \frac{F_A^2}{4m_a^2} - \frac{F_V^2}{4m_\rho^2}. \end{aligned} \quad (48)$$

The expressions for L_9 , L_{10} follow from the resonance saturation of the low-energy constants [17, 18].

The masses of the $\rho(770)$ and $a_1(1260)$ mesons in Eq. (46) are $m_\rho = 0.7755$ GeV and $m_a = 1.230$ GeV. The width of the $a_1(1260)$ meson in Eq. (46) is taken from Ref. [32]:

$$\begin{aligned} \Gamma_a(t) &= \Gamma_0 g(t)/g(m_a^2), \\ g(t) &= \left(1.623 t + 10.38 - \frac{9.23}{t} + \frac{0.65}{t^2}\right) \theta(t - (m_\rho + m_\pi)^2) \\ &\quad + 4.1(t - 9m_\pi^2)^3 [1.0 - 3.3(t - 9m_\pi^2) + 5.8(t - 9m_\pi^2)^2] \theta(t - 9m_\pi^2) \theta((m_\rho + m_\pi)^2 - t). \end{aligned} \quad (49)$$

It is implied in this equation that t is in GeV^2 , masses are in GeV ($m_\pi = m_{\pi^\pm}$) and all numbers are in appropriate powers of GeV. An alternative form of the off-mass-shell a_1 decay width is proposed in Ref. [33].

The values of the coupling constants F_A , F_V and G_V are presented in Table 1. The constants F_V and G_V are obtained from the experimental information [16] on the $\rho \rightarrow e^+e^-$ and $\rho \rightarrow \pi\pi$ decay widths: $\Gamma(\rho \rightarrow e^+e^-) = 7.04 \pm 0.06$ keV and $\Gamma(\rho \rightarrow \pi^+\pi^-) = 146.2 \pm 0.7$ MeV. To find F_V , G_V one can use the tree-level relations

$$\Gamma(\rho \rightarrow e^+e^-) = \frac{e^4 F_V^2}{12\pi m_\rho}, \quad \Gamma(\rho \rightarrow \pi^+\pi^-) = \frac{G_V^2 m_\rho^3}{48\pi F_\pi^4} \left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{3/2}.$$

The constant F_A is then calculated from Eq. (47) using the average value $f_A(0)_{exp} = 0.0119 \pm 0.0001$ measured in the radiative pion decays [16]. The constants F_V , G_V and F_A , calculated for central values of the data, are called hereafter “set 1” and are shown in Table 1.

As another option we choose theoretically motivated values of the constants from Ref. [18]. In particular, the relations $F_V = 2G_V$ and $F_V G_V = F_\pi^2$ are suggested there. The corresponding parameters are called “set 2” and are also given in Table 1.

In the calculation of the vector form factor $f_V(t)$ one needs the transition $W^\pm \rightarrow \rho^\pm \rightarrow \pi^\pm \gamma$, which involves an odd-intrinsic-parity (anomalous) vertex. For the latter we use the vector (or Proca) representation for the spin-one fields. As shown in Ref. [39] (see also

	F_A	F_V	G_V
set 1	0.1368 GeV	0.1564 GeV	0.06514 GeV
set 2	F_π	$\sqrt{2}F_\pi$	$F_\pi/\sqrt{2}$

Table 1. Two sets of the coupling constants. Values for set 1 are calculated from the $\rho \rightarrow e^+e^-$, $\rho \rightarrow \pi^+\pi^-$ decay widths and $f_A(0)_{exp}$ (see the text). Values for set 2 are chosen according to Ref. [18], and $F_\pi = 0.0924$ GeV.

Ref. [19]), the use of the vector field V^μ instead of the antisymmetric tensor field $V^{\mu\nu}$ in the description of the spin-one resonances ensures the correct behavior of the Green functions to order $\mathcal{O}(p^6)$, while the tensor formulation would require additional local terms (see also the discussion in Appendix F of Ref. [22]).

Thus we choose the Lagrangian [19, 39]

$$\mathcal{L}_{VP\gamma} = -h_V \epsilon_{\mu\nu\alpha\beta} \text{Tr} (V^\mu \{u^\nu, f_+^{\alpha\beta}\}) = \frac{4\sqrt{2}e h_V}{3F_\pi} \epsilon_{\mu\nu\alpha\beta} \partial^\alpha B^\beta \vec{\rho}^\mu \partial^\nu \vec{\pi} \quad (50)$$

with the coupling constant h_V .

For the $W^\pm \rightarrow \rho^\pm (K^{*\pm})$ vertex, in the vector formulation, one has

$$\begin{aligned} \mathcal{L}_{WV} &= -\frac{1}{4}g \frac{F_V}{m_\rho} \text{Tr} (W_{\mu\nu} \partial^\mu V^\nu) \\ &= -\frac{gF_V}{4m_\rho} W_{\mu\nu}^+ (V_{ud} \partial^\mu \rho^{-\nu} + V_{us} \partial^\mu K^{*- \nu}) + h.c. \end{aligned} \quad (51)$$

Using (50), (51) and adding the vertex (A.12), we obtain the vector form factor

$$f_V(t) = \frac{\sqrt{2}m_{\pi^\pm}}{F_\pi} \left[\frac{N_C}{24\pi^2} + \frac{4\sqrt{2}h_V F_V}{3m_\rho} \frac{t}{m_\rho^2 - t - im_\rho \Gamma_\rho(t)} \right]. \quad (52)$$

The width of the off-mass-shell ρ meson can be calculated from the interaction Lagrangian \mathcal{L}_{int}^R in Eqs. (A.9). It is written in the form

$$\Gamma_\rho(t) = \frac{G_V^2 m_\rho t}{48\pi F_\pi^4} \left[\left(1 - \frac{4m_\pi^2}{t}\right)^{3/2} \theta(t - 4m_\pi^2) + \frac{1}{2} \left(1 - \frac{4m_K^2}{t}\right)^{3/2} \theta(t - 4m_K^2) \right], \quad (53)$$

where $m_K = 0.4937$ GeV is the mass of the K^\pm meson. Other contributions to the width, coming, for example, from the four-pion decay of the ρ , are neglected in (53).

The coupling constant h_V can be fixed from the decay width $\Gamma(\rho^\pm \rightarrow \pi^\pm \gamma) = 68 \pm 7$ keV [16]. Then from the equation

$$\Gamma(\rho^\pm \rightarrow \pi^\pm \gamma) = \frac{e^2 m_\rho^3 h_V^2}{27\pi F_\pi^2} \left(1 - \frac{m_\pi^2}{m_\rho^2}\right)^3$$

we obtain $h_V = 0.036$. Alternatively, h_V can be constrained from the high-energy behavior of the vector form factor. Such constraints have been used in Refs. [9, 12]. According to the asymptotic predictions of the perturbative QCD [40], at $t \rightarrow -\infty$ the form factor behaves as $f_V(t) \sim \text{const}/t$. Imposing this condition on $f_V(t)$ in Eq. (52) one obtains

$$h_V = \frac{N_c m_\rho}{32\pi^2 \sqrt{2} F_V}. \quad (54)$$

This yields $h_V = 0.033$ (0.040) for F_V from the set 1 (set 2) in Table 1. These values are close to $h_V = 0.036$ derived from the $\rho \rightarrow \pi\gamma$ decay width, and the latter value is used in our calculations. Recently an approach based on Lagrangian with bilinear in resonances terms has been applied in Ref. [41], and the high-energy constraints in the anomalous QCD sector have been formulated in the tensor representation for the spin-one fields. Note that in this approach the vector form factor depends on several parameters which can be determined from these high-energy constraints.

5. CALCULATION OF DIFFERENTIAL DECAY WIDTH, STOKES PARAMETERS, POLARIZATION ASYMMETRY AND SPIN-CORRELATION PARAMETERS

5.1. The t -dependence in the case of unpolarized τ^-

Because the vector and axial-vector form factors depend on invariant mass of the $\pi-\gamma$ system, one can integrate the double differential width (spin-independent and spin-dependent) at fixed values of variable t , using the restrictions (7). For the decay width $d\Gamma_0$ we have

$$\frac{d\Gamma_0}{dt} = P [I_0(t) + (|a(t)|^2 + |v(t)|^2) A_0(t) + \text{Re}(a(t)) B_0(t) + \text{Re}(v(t)) C_0(t)], \quad P = \frac{Z^2}{2^8 \pi^3 M^2}, \quad (55)$$

where $I_0(t)$ is the contribution of the inner bremsstrahlung

$$I_0(t) = \frac{4M}{t - m^2} \left\{ \frac{M^2 - t}{t} [(t + m^2)^2 - 4M^2 t] + [2M^2(M^2 + t - m^2) - m^4 - t^2] L \right\}, \quad L = \ln \frac{M^2}{t}.$$

As one can see, from Eq. (55), the structure-dependent (resonance) contribution into $d\Gamma_0/dt$ does not contain vector-axial-vector interference, but it includes a sum of the squared moduli of the vector and axial-vector form factors. This sum is multiplied by the function

$$A_0(t) = \frac{(t - m^2)^3 (M^2 - t)^2 (M^2 + 2t)}{3M^5 t^2}.$$

The interference of the IB and structural amplitudes includes only real parts of the form factors and

$$B_0(t) = \frac{4(t - m^2)}{Mt} [(2t + M^2 - m^2)(M^2 - t) + t(m^2 - 2M^2 - t)L],$$

$$C_0(t) = \frac{4(t - m^2)^2}{Mt} (t - M^2 + tL).$$

As concerns the quantity $d\Gamma_1/dt$ connected with the Stokes parameter ξ_1 (see Eq. 14 and formulas just below), it reads

$$\frac{d\Gamma_1}{dt} = P [Im(a(t)^* v(t)) A_1(t) + Im(v(t)) C_1(t)], \quad (56)$$

where

$$A_1(t) = -2 \frac{(t - m^2)^3 (M^2 - t)^3}{3M^5 t^2}, \quad C_1(t) = \frac{4(t - m^2)}{Mt} (t^2 - M^4 + 2M^2 t L).$$

This quantity, in the case of an unpolarized τ , is the only one which includes the imaginary parts of the vector-axial-vector interference and imaginary part of the vector form factor. Because it does not contain the pure IB contribution, it may be useful to study the resonance one.

We have also

$$\frac{d\Gamma_2}{dt} = P [I_2(t) + Re(a(t)^* v(t)) A_2(t) + Re(a(t)) B_2(t) + Re(v(t)) C_2(t)], \quad (57)$$

where

$$I_2(t) = -\frac{4M}{t} [(3t + m^2)(M^2 - t) - t(t + m^2 + 2M^2)L],$$

$$A_2 = -2 A_0(t), \quad B_2(t) = -C_0(t), \quad C_2(t) = -B_0(t)$$

for the part corresponding to the circular polarization of the gamma quantum (parameter ξ_2) and

$$\frac{d\Gamma_3}{dt} = P [I_3(t) + (|a(t)|^2 - |v(t)|^2) A_3(t) + Re(a(t)) B_3(t)], \quad (58)$$

$$I_3(t) = \frac{8M(M^2 - m^2)}{t - m^2} (2(t - M^2) + (M^2 + t)L),$$

$$A_3(t) = -\frac{1}{2} A_1(t), \quad B_3(t) = -C_1(t)$$

for the part connected with parameter ξ_3 .

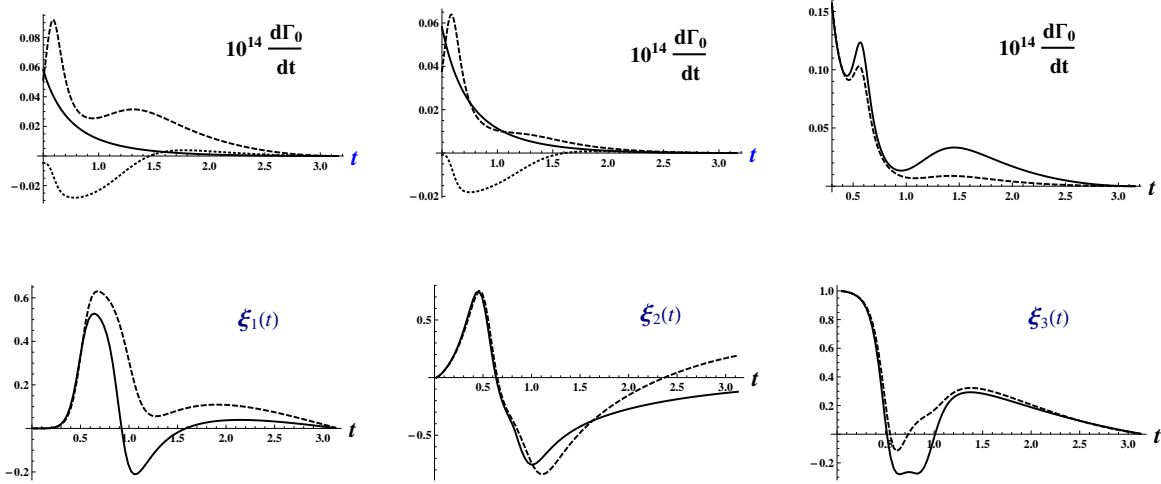


Figure 4. The t -distribution of the differential decay width, in GeV^{-1} , is shown in the upper row. The first (second) figure corresponds to the set 1 (set 2) of the resonance parameters given in the Table 1; the solid curve represents the inner bremsstrahlung contribution, the dashed one - resonance contribution and the dotted curve describes their interference. The third figure shows the sum of all the contributions: the solid (dashed) curve corresponds to the set 1 (set 2). The quantities ξ_i , (see Eq. (14)) in the lower row, are calculated including all the contributions for two sets of the parameters.

Obtained results are illustrated in Fig. 4 where we show the quantities $d\Gamma_0(t)/dt$ and $\xi_i(t)$ defined as

$$\xi_i(t) = \frac{d\Gamma_i(t)/dt}{d\Gamma_0(t)/dt}, \quad i = 1, 2, 3$$

5.2. The t -dependence for polarized τ^-

If τ^- , in the decay (1), is polarized, we can also write analytical expressions for the quantities

$$\frac{d\Gamma_0^S}{c_2 dc_2 dt}, \quad \frac{d\Gamma_i^S}{c_2 dc_2 dt}, \quad i = 1, 2, 3.$$

They can be obtained from the corresponding fully differential distributions by integration with respect to c_1 (using relations (9)) and ω at fixed t . Remind that in the rest frame of τ^- , its polarization 3-vector is directed along the Z axis and in this subsection we consider effects caused by the component of this 3-vector which belongs to the decay plane (\mathbf{q}, \mathbf{k}).

The quantity, which defines the polarization asymmetry of the decay, reads

$$\begin{aligned} \frac{d\Gamma_0^S}{c_2 dc_2 dt} &= \frac{P}{2} [I_0^S(t) + (|a(t)|^2 + |v(t)|^2)A_0^S(t) + \\ &Re(a(t)^*v(t))B_0^S(t) + Re(a(t))C_0^S(t) + Re(v(t))D_0^S(t)] , \end{aligned} \quad (59)$$

where

$$\begin{aligned} I_0^S(t) &= \frac{4M}{(t-m^2)} \left[-\frac{m^4 M^2}{t} + m^4 - 2m^2 M^2 - 6M^4 + 2m^2 t + 3M^2 t + 3t^2 + \right. \\ &\quad \left. + [(m^2 + M^2)^2 + (M^2 + t)^2 + 4M^2 t]L \right] , \\ A_0^S(t) &= \frac{(t-m^2)^3}{3M^5 t^2} [M^6 - 6M^4 t + 3M^2 t^2 + 2t^3 + 6M^2 t^2 L] , \quad B_0^S(t) = \frac{4(t-m^2)^3}{M^3 t} (M^2 - t - tL) , \\ C_0^S(t) &= \frac{4(m^2 - t)}{Mt} [(t - M^2)(M^2 + m^2 + 4t) + t(m^2 + 4M^2 + t)L] , \\ D_0^S(t) &= \frac{4(m^2 - t)}{Mt} [(M^2 - t)(m^2 + 3t) - t(m^2 + 2M^2 + t)L] . \end{aligned}$$

Quantities $d\Gamma_i^S$, $i = 1, 2, 3$ describe correlations between the polarization states of τ^- and photon. For them we have

$$\frac{d\Gamma_1^S}{c_2 dc_2 dt} = \frac{P}{2} [Im(a(t)^*v(t))B_1^S(t) + Im(a(t))C_1^S(t) + Im(v(t))D_1^S(t)] , \quad (60)$$

$$\begin{aligned} B_1^S(t) &= A_1(t) , \quad C_1^S(t) = \frac{8(t-m^2)}{M} [2(M^2 - t) - (M^2 + t)L] , \\ D_1^S(t) &= \frac{4(t-m^2)}{Mt} [(t - M^2)(5t + M^2) + 2t(2M^2 + t)L] ; \\ \frac{d\Gamma_2^S}{c_2 dc_2 dt} &= \frac{P}{2} [I_2^S(t) + (|a(t)|^2 + |v(t)|^2)A_2^S(t) + Re(a(t)^*v(t))B_2^S(t) + \\ &Re(a(t))C_2^S(t) + Re(v(t))D_2^S(t)] , \end{aligned} \quad (61)$$

$$\begin{aligned} I_2^S(t) &= -\frac{M^2}{t-m^2} D_0^S(t) \\ A_2^S(t) &= \frac{2(t-m^2)^3}{M^3 t} (t - M^2 + tL) , \quad B_2^S(t) = -2A_0^S(t) , \\ C_2^S(t) &= -D_0^S(t) , \quad D_2^S(t) = -C_0^S(t) ; \end{aligned}$$

$$\frac{d\Gamma_3^S}{c_2 dc_2 dt} = \frac{P}{2} [I_3^S(t) + (|a(t)|^2 - |v(t)|^2)A_3^S(t) + Re(a(t))C_3^S(t) + Re(v(t))D_3^S(t)] , \quad (62)$$

$$I_3^S(t) = \frac{8M}{(t-m^2)t} [-(M^2 - t)(t + 2m^2 + 3M^2) + (M^4 + 3M^2 t + m^2(t + M^2))L] ,$$

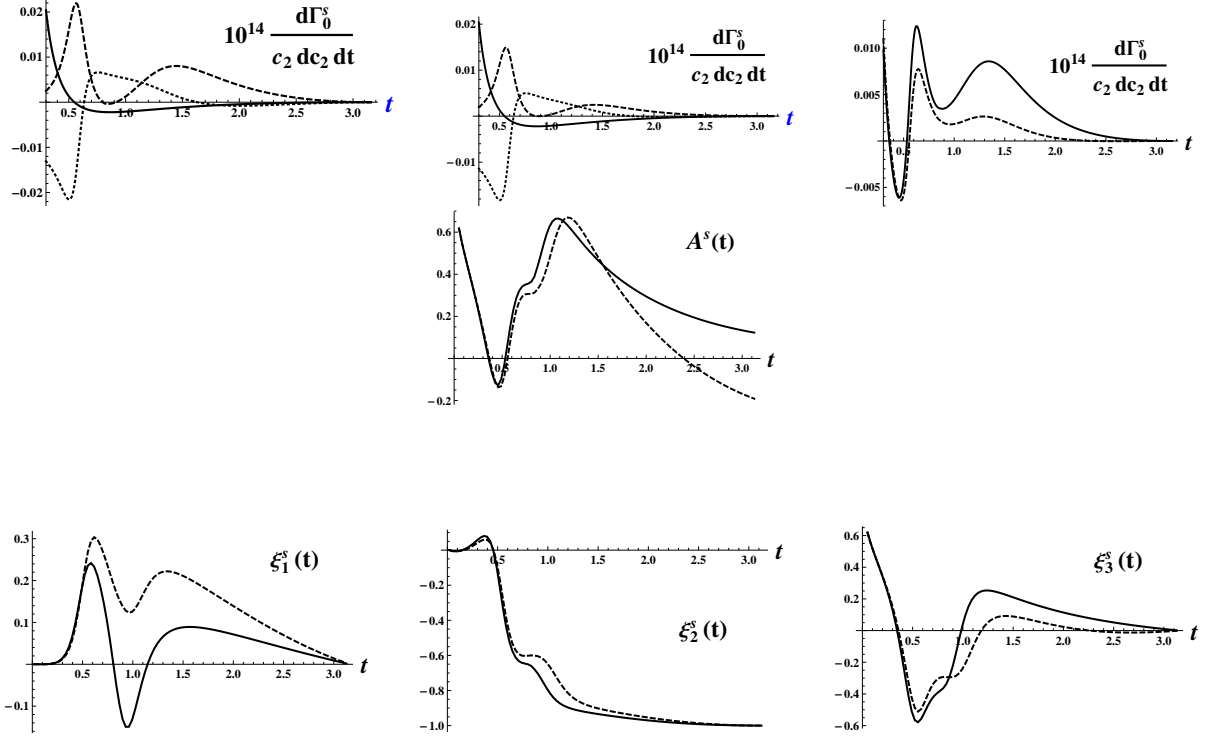


Figure 5. The notation of the curves in the figures in the upper and lower rows is the same as in Fig. 4. The figure in the middle row shows the quantity defined in accordance with Eq. (63) for two sets of the parameters. Quantities ξ_i^S are defined in Eq. (64).

$$A_3^S(t) = -\frac{1}{2}A_1(t), \quad C_3^S(t) = -D_1^S(t), \quad D_3^S(t) = -C_1^S(t).$$

It is easy to see that all the quantities and $d\Gamma_i^S$ vanish during the integration over the full angular region because they are proportional to $c_2 dc_2$.

In Fig. 5 we show the quantities: $d\Gamma_0^S/c_2 dc_2 dt$ that is the part of the differential decay width which depends on the τ^- polarization, the ratio

$$A^S(t) = \frac{2d\Gamma_0^S/(c_2 dc_2 dt)}{d\Gamma_0/dt} \quad (63)$$

about which we can say that $c_2 A^S(t)$ is the decay polarization asymmetry at fixed values of c_2 and t , as well as the parameters

$$\xi_i^S(t) = \frac{2d\Gamma_i^S/(c_2 dc_2 dt)}{d\Gamma_0/dt}, \quad i = 1, 2, 3, \quad (64)$$

which characterize different correlations between the polarization states of τ^- and γ quantum in the process (1).

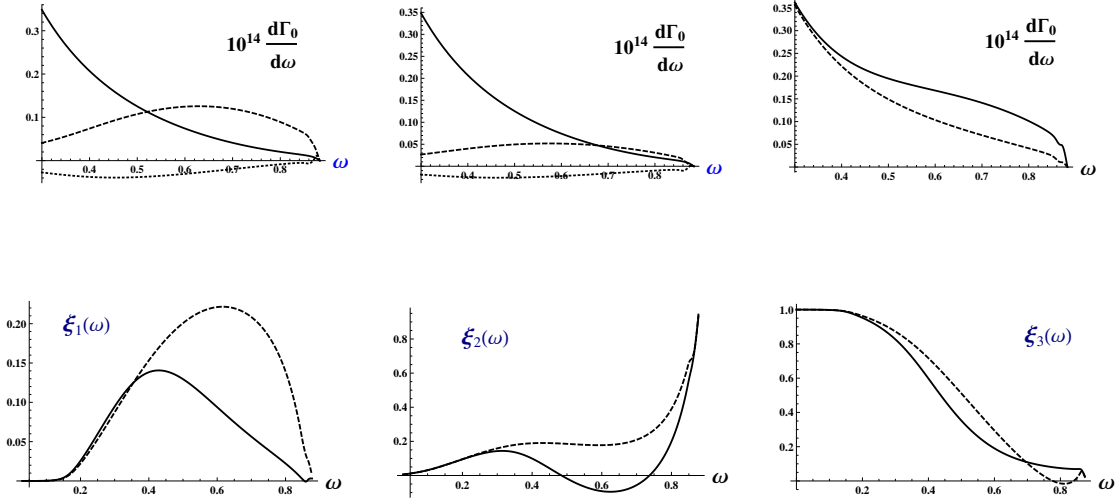


Figure 6. The photon spectrum in the decay (1) and the Stokes parameters versus the photon energy. Notation of the curves is the same as in Fig. 4.

5.3. Dependence on the photon energy

The photon energy distribution requires integration over the pion energy in the limits defined by the inequality (6). This integration cannot be performed analytically because of non trivial dependence of the vector and axial-vector form factors on the pion energy. In this section we illustrate the results of our numerical calculations for both unpolarized (Fig. 6) and polarized (Fig. 7) τ lepton.

In Figs. 6 and 7 we restrict the photon energy for the widths because at smaller energies the IB contribution becomes very large (due to infrared divergence) as compared with the structure-dependent one and their interference, and the corresponding part of the figures is not illustrative. As for the Stokes parameters, the expressiveness of the figures remains good for the small energies.

In the polarized case we define the quantities $A^S(\omega)$ and $\xi_i(\omega)$ in full analogy with the relations (63) and (64) for $A^S(t)$ and $\xi_i(t)$.

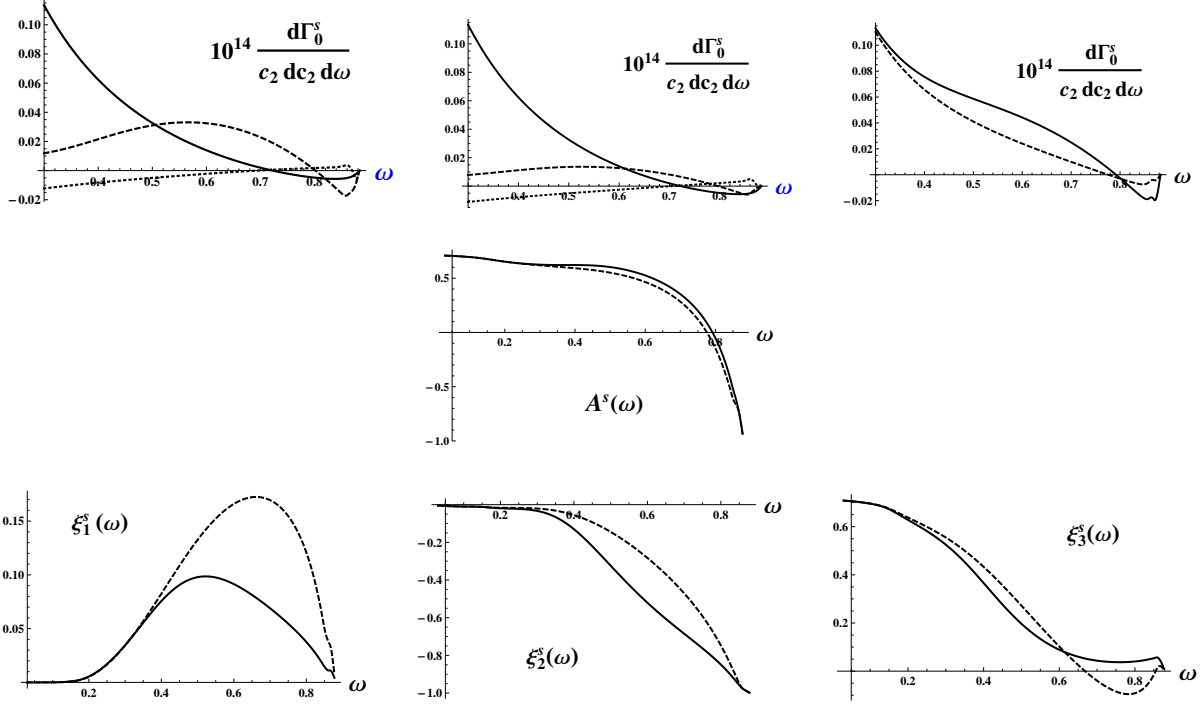


Figure 7. The spin-dependent part of the photon spectrum, the decay polarization asymmetry and the spin-correlation parameters as a function of the photon energy. Notation of the curves is the same as in Fig. 5.

6. DISCUSSION

To determine the moduli and phases of the form factors, a procedure was suggested in Ref. [10] (see Eq. (66)) that does not require the polarization measurements. For separation of contributions of different form factor combinations it was suggested to measure photon energy dependence of the differential probability $d\Gamma/d\omega dt$ at fixed t value (i.e., at a fixed value of the sum of the photon and pion energies). The obtained expression (in zero pion mass approximation) for this quantity is a sum of the terms multiplied by the photon energy to negative and positive powers. The measurement of this distribution permits, in principle, to find the coefficients in this power series. The following combinations of the form factors $Re v(t)$, $Re a(t)$, $|v(t)|^2 + |a(t)|^2$ and $Re a(t)v^*(t)$ can be determined from these coefficients. Note that our calculations are performed without neglecting the pion mass. This is important for the decay $\tau^- \rightarrow K^- \gamma \nu_\tau$ where it is necessary to take into account the kaon mass.

Eq. (56) shows that the Stokes parameter ξ_1 , as a function of the variable t , is determined by the imaginary parts of the vector and axial-vector form factors. But it follows from the

unitarity condition that $Imv(t) \neq 0$ for $t > 4m^2$ and $Ima(t) \neq 0$ for $t > 9m^2$. So, this parameter must be zero for $t < 4m^2$. Thus, the value of the parameter $\xi_1(t)$ is completely determined by the resonance contribution to the matrix element. The measurement of this parameter in the region $t > 4m^2$ can test the validity of this mechanism for the description of the decay (1). Of particular interest is region of the high values of the t variable where it may be necessary to take into account the contribution of the additional resonances beyond the ρ and a_1 mesons which are included in this paper. From Fig. 4 one can see that $\xi_1(t)$ parameter is sensitive to the choice of the parameters describing the resonance contribution. In the region $1GeV^2 < t < 1.5GeV^2$, the parameter $\xi_1(t)$ has opposite signs for the parameter set1 and set2. The same conclusions are valid for the spin correlation coefficient $\xi_1^s(t)$ (the Stokes parameter $\xi_1(t)$ which depends on the τ lepton polarization vector), as it is seen from Fig. 5. The Stokes parameters $\xi_1(\omega)$ and $\xi_1^s(\omega)$, as a function of the photon energy, is also sensitive to the choice of the parameter sets but in the region $t > 1.5GeV^2$. In this case the signs of the parameters $\xi_1(\omega)$ and $\xi_1^s(\omega)$ are the same for both parameter sets.

The Stokes parameter $\xi_3(t)$ contains the contributions of the IB, the interference between IB and resonance terms (which is determined by the $Rea(t)$) and resonance term which depends on the combination $|a(t)|^2 - |v(t)|^2$. So, this parameter is less sensitive to the choice of the parameter sets than the Stokes parameter $\xi_1(t)$. Nevertheless, the sizeable sensitivity exists in the region $0.5 GeV^2 < t < 1 GeV^2$. The Stokes parameter $\xi_3^s(t)$ is appreciably less sensitive to the choice of the parameter sets than the parameter $\xi_3(t)$ (Fig. 5). The Stokes parameter $\xi_3(\omega)(\xi_3^s(\omega))$, as a function of the photon energy, is also less sensitive than the parameter $\xi_1(\omega)(\xi_1^s(\omega))$.

Remind that the meaning of the parameters ξ_1 and ξ_3 requires the knowledge of the photon polarization vectors e_1 and e_2 . But to do this it is necessary to know the photon and pion momenta. As to the parameter ξ_2 , in this case it is sufficient to know only photon momentum.

The Stokes parameter $\xi_2(t)$ contains the contributions of the IB, the interference between IB and resonance terms (which is determined by the $Rea(t)$ and $Rev(t)$) and resonance term which depends on the $Re(a(t)v^*(t))$. From Fig. 4 one can see that this parameter is sensitive to the choice of the parameter sets in the region of the high values of the variable t ($t > 1 GeV^2$). The Stokes parameter $\xi_2^s(t)$ is weakly sensitive to the choice. The corresponding parameters, as a function of the photon energy, show the greater sensitivity to this choice in

comparison with the same parameters as functions of the t variable.

The photon energy has to be large enough to study the sensitivity to the choice of the model parameters. Although the number of events in this region is an order of magnitude smaller than in the low energy photon one, where the IB-contribution dominates, one can expect that high statistics precision measurements at the Super $c - \tau$ and SuperKEKB factories make it possible to improve some model resonance parameters used in our and in a number of other papers.

Note also that in Refs. [10, 31] the authors suggested to study the resonance mechanism in the radiative τ decay by selecting events (in the rest frame) at the maximal possible pion energy $\varepsilon = (M^2 + m^2)/(2M)$, where the contribution of IB to the decay width vanishes due to the radiative zeros of electromagnetic amplitudes for point-like particles [42]. But the corresponding number of events decreases due to the essential decrease of the kinematic region. On the other hand, one can be sure that at chosen direction of axes the IB contribution to the Stokes parameter $\xi_1(\varepsilon, \omega)$ vanishes in the whole kinematic region.

Most of the analytic calculations presented in this paper can also be used for analysis of the decay $\tau^- \rightarrow \nu_\tau K^- \gamma$. All the results in sections 2 and 3 will remain the same, apart from trivial changes of the masses, form factors and CKM matrix elements. Moreover, the t -distributions, obtained in section 5, will retain their form, but will be expressed in terms of the kaon mass and the corresponding vector and axial-vector form factors. The latter can be derived in framework of the R χ T following the procedure in section 4. We plan to perform calculations for the decay $\tau^- \rightarrow \nu_\tau K^- \gamma$ in the future.

7. CONCLUSION

We have investigated the radiative one-meson decay of the τ lepton, $\tau^- \rightarrow \pi^- \gamma \nu_\tau$. The photon energy spectrum and t -distribution (t is the square of the invariant mass of the pion-photon system) of the unpolarized τ lepton decay have been calculated. We have also studied the polarization effects in this decay. The following polarization observables have been calculated: the asymmetry caused by the τ lepton polarization, the Stokes parameters of the emitted photon and spin correlation coefficients which describe the influence of the τ lepton polarization on the photon Stokes parameters.

The amplitude of the τ lepton decay, $\tau^- \rightarrow \pi^- \gamma \nu_\tau$, has two contributions: the inner

bremsstrahlung which does not contain any free parameters and structure-dependent term which is parameterized in terms of the vector and axial-vector form factors. Note that in our case these form factors are the functions of the t variable and $t > 0$, i.e., we are in the time-like region. The form factors in this region are the complex functions and their full determination, i. e., not only their moduli but their phases as well, is non-trivial in this case. To do this it is necessary to perform the polarization measurements.

We calculated the unpolarized and polarized observables for two sets of the parameters describing the vector and axial-vector form factors. The numerical estimation shows that some polarization observables (the asymmetry and the Stokes parameters, especially ξ_1) can be effectively used for the discrimination between two parameter sets since these observables have opposite signs in some regions of the variable t or the photon energy.

Acknowledgement

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APPENDIX A. INTERACTIONS IN FRAMEWORK OF THE CHIRAL THEORY WITH RESONANCES

In this Appendix we outline the framework for a calculation of the form factors for the decay $\tau^- \rightarrow \nu_\tau \pi^- \gamma$. We use the formalism of the chiral theory with resonances (R χ T) suggested in Refs. [17, 18]. The corresponding Lagrangian can be written as

$$\mathcal{L}_{R\chi T} = \mathcal{L}_\chi^{(2)} + \mathcal{L}_R, \quad (\text{A.1})$$

where the $SU(3)_L \otimes SU(3)_R$ chiral Lagrangian in the order $\mathcal{O}(p^2)$ is

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle, \quad (\text{A.2})$$

$$\begin{aligned} u_\mu &= i[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger], \\ \chi_+ &= u^\dagger \chi u^\dagger + u \chi^\dagger u, \quad \chi = 2B_0(s + ip), \end{aligned} \quad (\text{A.3})$$

where $\langle \dots \rangle$ means the trace in the flavor space and F is the pion weak decay constant in the chiral limit.

The octet of the pseudoscalar mesons P with $J^P = 0^-$ is included in the matrix

$$u(\Phi) = \exp(i\Phi/\sqrt{2}F) = 1 + i\Phi/\sqrt{2}F - \Phi^2/4F^2 + \dots,$$

where Φ is

$$\Phi = \begin{pmatrix} \pi^0/\sqrt{2} + \eta_8/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta_8/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta_8/\sqrt{6} \end{pmatrix}. \quad (\text{A.4})$$

The masses of the pseudoscalar mesons enter into Eq. (A.2) via the quark mass matrix \mathcal{M}

$$\chi = 2B_0\mathcal{M} + \dots, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s), \quad (\text{A.5})$$

and the constant B_0 is expressed through the quark condensate $\langle \bar{q}q \rangle$

$$B_0 = -\frac{\langle \bar{q}q \rangle}{F^2} = \frac{M_{\pi^{0,\pm}}^2}{m_u + m_d} = \frac{M_{K^0}^2}{m_d + m_s} = \frac{M_{K^\pm}^2}{m_u + m_s} = \frac{3M_\eta^2}{m_u + m_d + 4m_s}. \quad (\text{A.6})$$

The value $\langle \bar{q}q \rangle \approx (-240 \pm 10 \text{ MeV})^3$ (at an energy scale $\mu = 1 \text{ GeV}$). In the limit of exact isospin symmetry $\chi = \text{diag}(m_\pi^2, m_\pi^2, 2m_K^2 - m_\pi^2)$ in terms of the masses of π meson, m_π , and K meson, m_K .

The interaction of the pseudoscalar mesons with the W_μ^\pm , Z_μ -bosons and electromagnetic field B_μ can be included via the external fields r_μ and l_μ as follows

$$\begin{aligned} r_\mu &= -eQB_\mu + g \frac{\sin^2 \theta_W}{\cos \theta_W} QZ_\mu, \\ l_\mu &= -eQB_\mu - \frac{g}{2\sqrt{2}} W_\mu + \frac{g}{\cos \theta_W} (Q \sin^2 \theta_W + \frac{1}{6} - Q)Z_\mu. \end{aligned} \quad (\text{A.7})$$

Here the quark charge matrix is $Q = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$, $e = \sqrt{4\pi\alpha}$ is the positron charge, $g = e/\sin \theta_W$ is the $SU(2)_L$ coupling constant, and θ_W is the weak angle.

We also introduced the notation

$$W_\mu \equiv W_\mu^+ T_+ + W_\mu^- T_-, \quad (\text{A.8})$$

with $W_\mu^\pm = (W_1 \mp W_2)_\mu$ being the field of the charged weak bosons, and matrices T_+ , T_- are defined as

$$T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_- = T_+^\dagger,$$

where V_{ud} and V_{us} are the elements of the CKM matrix [27].

The lowest-order even-intrinsic-parity Lagrangian \mathcal{L}_R , describing interaction of the resonance fields with the pseudoscalars, has been suggested in Ref. [17]. It is linear in the resonance fields. In this Lagrangian we keep the contributions from the vector and axial-vector mesons relevant for the process of τ decay to π and γ :

$$\begin{aligned}\mathcal{L}_R &= \mathcal{L}_{kin}^R + \mathcal{L}_{int}^R, \\ \mathcal{L}_{kin}^R &= -\frac{1}{2} \sum_{R=V,A} \left\langle \nabla^\lambda R_{\lambda\mu} \nabla_\nu R^{\nu\mu} - \frac{M_R^2}{2} R_{\nu\mu} R^{\nu\mu} \right\rangle, \\ \mathcal{L}_{int}^R &= \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{iG_V}{\sqrt{2}} \langle V_{\mu\nu} u^\mu u^\nu \rangle + \frac{iF_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle,\end{aligned}\tag{A.9}$$

where F_V , G_V , F_A are the coupling constants, and antisymmetric tensor representation for the spin-1 fields $V_{\mu\nu}$ and $A_{\mu\nu}$ is applied [17].

Also here

$$\begin{aligned}f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, & F_R^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \\ F_L^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \\ \nabla_\mu X &= \partial_\mu X + [\Gamma_\mu, X], & \Gamma_\mu &= 1/2 \{u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\}.\end{aligned}\tag{A.10}$$

The fields $R = \{V, A\}$ represent nonets (octet and singlet) of the vector and axial-vector resonances with the lowest masses. In general, the higher-lying multiplets can be added if needed. However, already the low-lying resonances saturate the low-energy constants of χ PT [17]. We do not include here interactions nonlinear in the resonance fields; the structure of such terms has been investigated in Refs. [19, 34, 35].

For description of the vector form factor one needs the interaction Lagrangian in the odd-intrinsic-parity sector. The lowest order $\mathcal{O}(p^4)$ interactions follow from the Wess-Zumino-Witten (WZW) functional [36]. It is sufficient to keep there only the terms which are linear in the pseudoscalar fields Φ and containing two external fields l_μ , r_μ . Thus we retain

$$\begin{aligned}\mathcal{L}_{WZW}^{(2)} &= -\frac{\sqrt{2}N_C}{48\pi^2 F} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[\partial_\mu \Phi \left(\frac{1}{2} l_\sigma \partial_\nu r_\rho + \frac{1}{2} r_\sigma \partial_\nu l_\rho + \frac{1}{2} r_\nu \partial_\rho l_\sigma + \frac{1}{2} l_\nu \partial_\rho r_\sigma + \right. \right. \\ &\quad \left. \left. + l_\nu \partial_\rho l_\sigma + r_\nu \partial_\rho r_\sigma + l_\sigma \partial_\nu l_\rho + r_\sigma \partial_\nu r_\rho \right) \right]\end{aligned}\tag{A.11}$$

where $N_C = 3$ is the number of the quark colors, and $\epsilon^{0123} = +1$.

The external fields r_μ , l_μ are further expressed through the electromagnetic field and the field of the W boson in Eq. (A.7). Choosing in Eq. (A.11) only the electromagnetic field

one would obtain $\pi^0\gamma\gamma$, $\eta\gamma\gamma$ and $\eta'\gamma\gamma$ interactions. Here we are interested in the terms proportional to both fields, of the photon and W -boson:

$$\begin{aligned}\mathcal{L}_{\gamma W\Phi} &= \frac{3egN_C}{48\pi^2 F} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[\Phi \left(\partial_\mu B_\nu Q \partial_\rho W_\sigma + \partial_\rho W_\sigma Q \partial_\mu B_\nu \right) \right] \\ &= \frac{egN_C}{48\pi^2 F} \epsilon^{\mu\nu\rho\sigma} \partial_\mu B_\nu \left[V_{ud}(\pi^- \partial_\rho W_\sigma^+ + \pi^+ \partial_\rho W_\sigma^-) + V_{us}(K^- \partial_\rho W_\sigma^+ + K^+ \partial_\rho W_\sigma^-) \right].\end{aligned}\quad (\text{A.12})$$

An additional odd-intrinsic-parity interaction relevant for the transition $W \rightarrow V \rightarrow P\gamma$ is considered in Section 4.

From the Lagrangians (A.1), (A.11) and (A.12) one can obtain the necessary terms describing interactions of the pseudoscalar mesons, resonances, W boson and photon.

APPENDIX B. POLARIZATION OF τ^- LEPTON IN ELECTRON-POSITRON ANNIHILATION

Let us consider the polarization of the τ^- lepton in annihilation process

$$e^-(p_1) + e^+(p_2) \rightarrow \tau^-(q_1) + \tau^+(q_2)$$

provided the electron has nonzero longitudinal polarization. The effect can be understood by means of the corresponding matrix element squared

$$\begin{aligned}|M|^2 &= E^{\mu\nu} T_{\mu\nu}, \quad \frac{1}{2} E^{\mu\nu} = -\frac{s}{2} g^{\mu\nu} + (p_1 p_2)^{\mu\nu} + i\lambda(\mu\nu p_1 p_2), \\ \frac{1}{2} T_{\mu\nu} &= -\frac{s}{2} g^{\mu\nu} + (q_1 q_2)^{\mu\nu} + iM(\mu\nu q S), \quad q = p_1 + p_2 = q_1 + q_2, \quad q^2 = s,\end{aligned}\quad (\text{B.1})$$

where S is the τ polarization 4-vector, and λ is the electron polarization degree.

The contraction of the tensors in Eq. (B1) reads (neglecting the electron mass)

$$|M|^2 = A + B(S), \quad A = s^2 \left[1 + \cos^2 \theta + \frac{4M^2}{s} \sin^2 \theta \right], \quad B(S) = -4\lambda M s [(p_1 S) - (p_2 S)], \quad (\text{B.2})$$

where θ is the angle between the momenta of the electron and τ^- lepton.

The polarizations of τ^- are defined as follows [30]

$$P^L = \frac{B(S^L)}{A}, \quad P^T = \frac{B(S^T)}{A}. \quad (\text{B.3})$$

If we choose the 4-vectors S^L and S^T in such a way that in the rest frame of the τ^- lepton they are

$$S^L = (0, \mathbf{n}_1), \quad S^T = (0, \mathbf{n}_2), \quad \mathbf{n}_1 \mathbf{n}_2 = 0, \quad \mathbf{n}_1^2 = \mathbf{n}_2^2 = 1, \quad (\text{B.4})$$

where \mathbf{n}_1 is in the direction of the 3-vector \mathbf{q}_1 in c.m.s. and \mathbf{n}_2 belongs to the scattering plane, then we can write the covariant form of the 4-vectors S^L and S^T in terms of the momentum 4-vectors, namely (neglecting the electron mass)

$$\begin{aligned} S^L &= \frac{(q_1 q_2) q_1 - M^2 q_2}{M \sqrt{(q_1 q_2)^2 - M^4}}, \\ S^T &= \frac{[M^2 - (p_2 q_1)] q_1 + [M^2 - (p_1 q_1)] q_2 + [(p_1 p_2) - 2M^2] p_1}{N}, \\ N^2 &= [(p_1 p_2) - 2M^2][2(q_1 p_1)(q_1 p_2) - M^2(p_1 p_2)]. \end{aligned} \quad (\text{B.5})$$

It is easy to show that in the rest frame of τ^- covariant forms in Eq. (B5) coincides with relations given in Eq. (B4). Using the above formulae one finds

$$B(S^L) = 2\lambda s^2 \cos \theta, \quad B(S^T) = 4\lambda M s \sqrt{s} \sin \theta \quad (\text{B.6})$$

and the corresponding results for the polarizations P^L and P^T are given in the text.

At the Super $c-\tau$ factory planned in Novosibirsk, τ -pairs will be created near the threshold where the directions of their 3-momenta are not determined, therefore above evaluations are not convenient. The only marked direction, for the considered reaction in this case in *c.m.s.*, is the colliding beams one. It means that choosing in Eq. (B4) the unit 3-vector \mathbf{n}_1 along the electron beam direction \mathbf{p}_1 and, as before, \mathbf{n}_2 is in the reaction plane, we are able to go to the threshold limit and clear interpret the τ^- polarization states.

This requires a modification of the corresponding covariant forms for the polarization 4-vectors as follows

$$\begin{aligned} S^l &= \frac{1}{M} \left(\frac{M^2}{(p_1 q_1)} p_1 - q_1 \right), \\ S^t &= \frac{1}{N_1} \left[\left(\frac{M^2}{(p_1 q_1)} - 1 + \frac{(p_1 q_1)}{(p_1 p_2)} \right) p_1 + \frac{(p_1 q_1)}{(p_1 p_2)} p_2 - q_1 \right], \\ N_1^2 &= 2(p_1 q_1) - M^2 - \frac{2(p_1 q_1)^2}{(p_1 p_2)}. \end{aligned} \quad (\text{B.7})$$

In this case we have

$$\begin{aligned} B(S^{l,t}) &= 2\lambda s^2 D^{l,t}, \quad D^l = \frac{4M^2}{s} \left[1 - \sqrt{1 - \frac{4M^2}{s}} \cos \theta \right]^{-1} - \sqrt{1 - \frac{4M^2}{s}} \cos \theta, \\ D^t &= -\frac{2M}{\sqrt{s}} \sqrt{1 - \frac{4M^2}{s}} \sin \theta, \end{aligned}$$

and near the threshold when

$$1 - \frac{4M^2}{s} \ll 1$$

we have

$$P^l = \lambda \left[1 - \frac{1}{2} \left(1 - \frac{4M^2}{s} \right) \sin^2 \theta \right] \quad P^t = -\frac{2\lambda M \sin \theta}{\sqrt{s}} \sqrt{1 - \frac{4M^2}{s}}. \quad (\text{B.8})$$

Just at the threshold ($s = 4M^2$) $P^l = \lambda$, $P^t = 0$.

Thus we see that near the threshold the τ lepton practically keeps the longitudinal polarization of the electron beam.

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